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Geometrical aspects of the two-component neutrino field in general relativity

J. B. GRIFFITHS and R. A. NEWING

Department of Applied Mathematics, University College of North Wales, Bangor, Caerns, Wales

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Abstract. Conditions for the existence of a neutrino field are expressed in terms of spin coefficients. It is shown that a neutrino-gravitational field with positive energy density necessarily defines a null geodesic congruence. The structure of the Ricci tensor is examined and shown to be closely related to the twist and shear of the congruence.

1. Introduction

It is well known that combined gravitational and electromagnetic fields can be given a geometrical interpretation in general relativity. A completely geometrical theory of the two-component neutrino-gravitational field, however, has not yet been established. In our paper (Griffiths and Newing 1970 to be referred to as I) Weyl's neutrino equations were expressed as equivalent tetrad equations, and these equations are now expressed as conditions on the spin coefficients (Newman and Penrose 1962) associated with the tetrad. The neutrino conditions, together with the requirement of a positive energy density, are shown to imply that the neutrino flux vector defines a null geodesic congruence, and twist-free neutrino congruences are found to be necessarily shear-free. The positive energy condition is a somewhat severe restriction for a fermion field, but the condition defines a physically interesting subclass of possible neutrino fields. The eigenvectors and eigenvalues of the neutrino energy-momentum tensor, and hence of the Ricci tensor of the space-time associated with the neutrino-gravitational field, are investigated, and the general nature of these tensors is found to be closely related to the shear and twist of the congruence.

2. Spin coefficients

We define a tetrad of null vectors, l_μ, n_μ, m_μ, $\bar{m}_\mu$ as in I. The components of l_μ and n_μ being real, and those of m_μ complex. The most general transformations on the tetrad, subject only to the restriction that the direction of l_μ remains unchanged, are the null rotations about l_μ:

$$l_\mu = \lambda^2 l_\mu' \qquad n_\mu = \lambda^{-2}(n_\mu' + \bar{\psi} m_\mu' + \psi \bar{m}_\mu' + \psi\bar{\psi} l_\mu')$$
$$m_\mu = \mathrm{e}^{2\mathrm{i}\eta}(m_\mu' + \psi l_\mu')$$

where the parameters λ and η are real, and ψ is complex. The tetrad may be defined in terms of two basis spinors ξ_A and χ_A such that $\xi_A\chi^B - \chi_A\xi^B = \delta_A{}^B$, and the corresponding spinor transformations are

$$\xi_A = \lambda \mathrm{e}^{-\mathrm{i}\eta}\xi_A' \qquad \chi_A = \lambda^{-1}\mathrm{e}^{\mathrm{i}\eta}(\chi_A' + \psi\xi_A').$$

Under these transformations the spin coefficients transform as follows:

$$
\begin{aligned}
\kappa &= l_{\alpha;\beta}m^{\alpha}l^{\beta} &&= \lambda^{4}\mathrm{e}^{2\mathrm{i}\eta}\kappa' \\
\rho &= l_{\alpha;\beta}m^{\alpha}\bar{m}^{\beta} &&= \lambda^{2}(\rho'+\bar{\psi}\kappa') \\
\sigma &= l_{\alpha;\beta}m^{\alpha}m^{\beta} &&= \lambda^{2}\mathrm{e}^{4\mathrm{i}\eta}(\sigma'+\psi\kappa') \\
\tau &= l_{\alpha;\beta}m^{\alpha}n^{\beta} &&= \mathrm{e}^{2\mathrm{i}\eta}(\tau'+\psi\rho'+\bar{\psi}\sigma'+\psi\bar{\psi}\kappa') \\
\epsilon &= \tfrac{1}{2}(l_{\alpha;\beta}n^{\alpha}+\bar{m}_{\alpha;\beta}m^{\alpha})l^{\beta} &&= \lambda^{2}\{\epsilon'+\bar{\psi}\kappa'+[\ln(\lambda\mathrm{e}^{\mathrm{i}\eta})]_{,\nu}l'^{\nu}\} \\
\alpha &= \tfrac{1}{2}(l_{\alpha;\beta}n^{\alpha}+\bar{m}_{\alpha;\beta}m^{\alpha})\bar{m}^{\beta} &&= \mathrm{e}^{-2\mathrm{i}\eta}\{\alpha'+\bar{\psi}\epsilon'+\bar{\psi}\rho'+\bar{\psi}^{2}\kappa'+[\ln(\lambda\mathrm{e}^{\mathrm{i}\eta})]_{,\nu}(\bar{m}'^{\nu}+\bar{\psi}l'^{\nu})\} \\
\beta &= \tfrac{1}{2}(l_{\alpha;\beta}n^{\alpha}+\bar{m}_{\alpha;\beta}m^{\alpha})m^{\beta} &&= \mathrm{e}^{2\mathrm{i}\eta}\{\beta'+\psi\epsilon'+\bar{\psi}\sigma'+\psi\bar{\psi}\kappa'+[\ln(\lambda\mathrm{e}^{\mathrm{i}\eta})]_{,\nu}(m'^{\nu}+\psi l'^{\nu})\} \\
\gamma &= \tfrac{1}{2}(l_{\alpha;\beta}n^{\alpha}+\bar{m}_{\alpha;\beta}m^{\alpha})n^{\beta} &&= \lambda^{-2}\{\gamma'+\psi\alpha'+\bar{\psi}\beta'+\psi\bar{\psi}\epsilon'+\psi(\tau'+\psi\rho'+\bar{\psi}\sigma'+\psi\bar{\psi}\kappa') \\
& && \quad +[\ln(\lambda\mathrm{e}^{\mathrm{i}\eta})]_{,\nu}(n'^{\nu}+\bar{\psi}m'^{\nu}+\psi\bar{m}'^{\nu}+\psi\bar{\psi}l'^{\nu})\}.
\end{aligned}
$$

If l_{μ} is the tangent vector of a null geodesic congruence then $l_{[\lambda}l_{\mu];\alpha}l^{\alpha} = 0$ and the necessary and sufficient condition for l_{μ} to be geodesic is $\kappa = 0$. This condition is invariant with respect to null rotations of the tetrad, and in this case the coefficients ρ, σ and ϵ are invariant with respect to ψ rotations.

If l_{μ} is geodetic, it is possible to introduce a null frame $\underset{\circ}{l}_{\mu}$, $\underset{\circ}{n}_{\mu}$, $\underset{\circ}{m}_{\mu}$ such that

$$\underset{\circ}{l}_{\mu;\alpha}\underset{\circ}{l}^{\alpha} = 0 \qquad \underset{\circ}{n}_{\mu;\alpha}\underset{\circ}{l}^{\alpha} = 0 \qquad \underset{\circ}{m}_{\mu;\alpha}\underset{\circ}{l}^{\alpha} = 0$$

and the corresponding spin coefficients have geometrical interpretations (see Jordan *et al.* 1961), for example, $|\underset{\circ}{\sigma}|$ is the 'shear', and if we put $\underset{\circ}{\rho} = \underset{\circ}{\theta}+\mathrm{i}\underset{\circ}{\omega}$, then $\underset{\circ}{\theta}$ is the 'expansion' and $\underset{\circ}{\omega}$ the 'twist'.

3. Neutrino field conditions

With the notation of I a given space–time will admit a two-component neutrino field if a null tetrad $(l_{\mu}, n_{\mu}, m_{\mu}, \bar{m}_{\mu})$ can be constructed satisfying the neutrino equation

$$S_{\mu;\nu}^{\ \ \nu} = H_{\mu} \tag{3.1}$$

and the gravitational equations

$$R_{\mu\nu}+E_{\mu\nu} = 0$$

where $S_{\mu\nu}$ is the self dual tensor $2l_{[\mu}m_{\nu]}$ and

$$E_{\mu\nu} = 2\mathrm{i}\{H_{(\mu}\bar{m}_{\nu)}-\bar{H}_{(\mu}m_{\nu)}+P_{(\mu}l_{\nu)}\}$$

where $H_{\mu} = m^{\alpha}l_{\alpha;\mu}$ and $P_{\mu} = \bar{m}^{\alpha}m_{\alpha;\mu}$. In terms of the spin coefficients associated with this tetrad

$$H_{\mu} = \tau l_{\mu}+\kappa n_{\mu}-\rho m_{\mu}-\sigma\bar{m}_{\mu}$$

and (3.1) is equivalent to

$$\epsilon = \rho \qquad \beta = \tau.$$

Using this latter result and putting $\rho = \theta+\mathrm{i}\omega$, $E_{\mu\nu}$ may be expressed in the form

$$
\begin{aligned}
E_{\mu\nu} = 2\{&-Al_{\mu}l_{\nu}+Bl_{(\mu}m_{\nu)}+\bar{B}l_{(\mu}\bar{m}_{\nu)}+2\omega l_{(\mu}n_{\nu)}+2\omega m_{(\mu}\bar{m}_{\nu)} \\
&+\mathrm{i}\bar{\sigma}m_{\mu}m_{\nu}-\mathrm{i}\sigma\bar{m}_{\mu}\bar{m}_{\nu}+\mathrm{i}\kappa n_{(\mu}\bar{m}_{\nu)}-\mathrm{i}\bar{\kappa}n_{(\mu}m_{\nu)}\}
\end{aligned}
$$

where $A = \mathrm{i}(\gamma-\bar{\gamma})$ and $B = \mathrm{i}(\alpha-2\bar{\tau})$.

Positive energy density is ensured by the condition

$$E_{\mu\nu}V^{\mu}V^{\nu} > 0 \tag{3.2}$$

where V^{μ} is an arbitrary real time-like vector. Taking this in the form

$$V_{\mu} = l_{\mu} + bn_{\mu} - cm_{\mu} - \bar{c}\bar{m}_{\mu} \qquad b > |c|^2$$

it follows from (3.2) that

$$b(-Ab+2\omega)+2|c|^2\omega+\mathrm{i}(\bar{c}^2\bar{\sigma}-c^2\sigma)+b(\bar{c}B+c\bar{B})+\mathrm{i}(c\kappa-\bar{c}\bar{\kappa}) > 0. \tag{3.3}$$

Considering V^{μ} with $c = 0$, we have either $\omega \geqslant 0$, $A < 0$ or $\omega > 0$, $A \leqslant 0$. Now for vectors with $c \neq 0$, put $b = \nu|c|^2$ and divide by $|c|^2$. Then (3.3) is satisfied if

$$\nu(-\nu|c|^2A+2\omega)+2\omega-2|\sigma|-2\nu|c||B|-2|\kappa|/|c| > 0 \tag{3.4}$$

for arbitrary values of the independent quantities ν and $|c|$, subject only to the condition $\nu > 1$. This immediately implies that $\kappa = 0$. Thus a neutrino field with positive energy density is necessarily geodesic, with the ray vector defining the direction of the neutrino flux. The three quantities θ, ω and $|\sigma|$ are then proportional to $\mathring{\theta}$, $\mathring{\omega}$ and $|\mathring{\sigma}|$—that is, to the expansion, twist and shear.

It may be verified that the neutrino equations and the expression for $E_{\mu\nu}$ are invariant with respect to ψ-transformations of the null tetrad. This fact is immediately obvious from the corresponding spinor equations:

$$\sigma^{\alpha\dot{A}B}\xi_{B|\alpha} = 0 \qquad \text{and} \qquad E_{\mu\nu} = 2\mathrm{i}\{\sigma_{(\mu A\dot{B}}\xi^{A}{}_{|\nu)}\xi^{\dot{B}} - \sigma_{(\mu A\dot{B}}\xi^{\dot{B}}{}_{|\nu)}\xi^{A}\}.$$

Hence, given the neutrino flux vector l_{μ}, we can introduce the transformation $m_{\mu} = m_{\mu}' + \psi l_{\mu}'$ and choose ψ in any convenient way. Now, under this transformation,

$$-\mathrm{i}B = \alpha - 2\bar{\tau} = -\mathrm{i}B' + 4\mathrm{i}\omega\bar{\psi} - 2\bar{\sigma}\psi$$

and hence ψ may be chosen to make B' zero unless the twist ω and the shear $|\sigma|$ are simultaneously zero, or $\omega = \frac{1}{2}|\sigma|$. However, in the cases where $\sigma = 0 = \omega$, or $\omega = \frac{1}{2}|\sigma|$, (3.4) requires that

$$-\nu|c|A-2|B| > 0 \qquad \text{or} \qquad -\nu|c|A+\frac{\nu-1}{\nu}\frac{|\sigma|}{|c|}-2|B| > 0$$

which are satisfied for all $\nu > 1$ and for all $|c|$ only if $B = 0$, since we have already shown that A cannot be positive. Equation (3.4), however, is slightly more restrictive than (3.3), but all the above results are implied by (3.3) except for the very special case where

$$A \neq 0 \qquad \omega = \tfrac{1}{2}|\sigma| \neq 0 \qquad \sigma B^2 = \mathrm{i}|\sigma||B|^2.$$

In this case (3.3) implies $|B|^2+4A|\sigma| \leqslant 0$ so that B is not necessarily zero.

Thus for positive energy density the energy–momentum tensor for the neutrino field can in general be put in the form

$$E_{\mu\nu} = -2Al_{\mu}l_{\nu}+2\omega(l_{\mu}n_{\nu}+n_{\mu}l_{\nu}+m_{\mu}\bar{m}_{\nu}+\bar{m}_{\mu}m_{\nu})+2\mathrm{i}\bar{\sigma}m_{\mu}m_{\nu}-2\mathrm{i}\sigma\bar{m}_{\mu}\bar{m}_{\nu}. \tag{3.5}$$

Returning now to the general case $\omega \neq 0$, (3.4) becomes

$$-\nu bA+2(\nu+1)\omega-2|\sigma| > 0$$

which implies the condition that

$$\omega \geqslant \tfrac{1}{2}|\sigma|. \tag{3.6}$$

This immediately implies that if a neutrino field with positive energy density is twist-free it must also be shear-free.† And so if a neutrino field is twist-free it must necessarily be a 'pure radiation field' discussed in I and defined by

$$E_{\mu\nu} = -2Al_\mu l_\nu \qquad A < 0. \tag{3.7}$$

Thus the condition that a given space–time shall admit a neutrino field with positive energy density is that there shall exist a tetrad such that the negative Ricci tensor is of the form (3.5), the null frame being such that the associated spin coefficients have the restrictions

$$\kappa = 0 \qquad \epsilon = \rho \qquad \beta = \tau \qquad \alpha = 2\bar{\tau}$$
$$\omega \geqslant \tfrac{1}{2}|\sigma| \qquad \mathrm{i}(\gamma - \bar{\gamma}) \leqslant 0.$$

A particular class of neutrino fields, which requires that $H_\mu = 0$, has been described by Penney (1965). This restriction requires the conditions $\sigma = 0$, $\kappa = 0$, $\theta = 0$, $\omega = 0$ and $\tau = 0$ and the neutrino conditions require further that there exists a null tetrad whose spin coefficients α, β and ϵ are also zero. Then for this tetrad $l_{\alpha;\beta}$ is of the form

$$l_{\alpha;\beta} = (\gamma + \bar{\gamma})l_\alpha l_\beta.$$

The further restriction which puts $l_{\alpha;\beta} = 0$ gives the 'restricted class' of neutrinos which is discussed by Inomata and McKinley (1965).

4. Eigenvalues and eigenvectors

The eigenvalues Λ and the eigenvectors V_μ of the energy–momentum tensor are such that

$$E_\mu{}^\alpha V_\alpha = \Lambda V_\mu.$$

For the neutrino field with positive energy density $E_\mu{}^\alpha$ is of the form (3.5) and its characteristic equation implies that, in general, it possesses three distinct real eigenvalues

$$\Lambda_1 = \Lambda_2 = 2\omega \qquad \Lambda_3 = -2\omega + 2|\sigma| \qquad \Lambda_4 = -2\omega - 2|\sigma|. \tag{4.1}$$

We have:

$$E_\mu{}^\alpha l_\alpha = 2\omega l_\mu$$
$$E_\mu{}^\alpha m_\alpha = -2\omega m_\mu + 2\mathrm{i}\sigma\bar{m}_\mu$$
$$E_\mu{}^\alpha n_\alpha = 2\omega n_\mu - 2Al_\mu.$$

Thus it is obvious that there exists a null eigenvector

$$V_\mu(1) = l_\mu$$

corresponding to the double eigenvalue $\Lambda = 2\omega$. There also exist two space-like eigenvectors corresponding to Λ_3 and Λ_4:

$$V_\mu(3) = (1-\mathrm{i})\sqrt{(\bar{\sigma})}m_\mu + (1+\mathrm{i})\sqrt{(\sigma)}\bar{m}_\mu$$
$$V_\mu(4) = (1+\mathrm{i})\sqrt{(\bar{\sigma})}m_\mu + (1-\mathrm{i})\sqrt{(\sigma)}\bar{m}_\mu.$$

† Similar results have been obtained independently by J. Wainwright in a forthcoming paper entitled *Geometric properties of neutrino fields in curved space–time*.

In the case where $\sigma = 0$, $\omega \neq 0$ we have a second repeated eigenvalue $\Lambda = -2\omega$. However, there still exist two real space-like eigenvectors given by

$$V_\mu(3) = m_\mu + \bar{m}_\mu$$
$$V_\mu(4) = \mathrm{i}(m_\mu - \bar{m}_\mu).$$

A fourth eigenvector may also exist corresponding to the repeated eigenvalue $\Lambda = 2\omega$. This must have the form

$$V_\mu(2) = Yn_\mu + Zm_\mu + \bar{Z}\bar{m}_\mu$$

and must satisfy the conditions

$$AY = 0 \qquad 2\omega Z = -\mathrm{i}\bar{\sigma}\bar{Z}. \tag{4.2}$$

If $A = 0$, then we can take $Z = 0$ and the fourth eigenvector is

$$V_\mu(2) = n_\mu.$$

In this case we have two null eigenvectors corresponding to $\Lambda = 2\omega$. These are equivalent to a time-like and a space-like eigenvector.

If $A \neq 0$, then (4.2) implies that a fourth eigenvector may exist corresponding to $Y = 0$, $Z \neq 0$, provided

$$\omega = \tfrac{1}{2}|\sigma|.$$

This case is just permitted by (3.6), but we can see from (4.1) that this corresponds to a thrice repeated eigenvalue

$$\Lambda_1 = \Lambda_2 = \Lambda_3 = 2\omega \qquad \Lambda_4 = -6\omega$$

and (4.2) implies that $V_\mu(2) = V_\mu(3)$.

Hence we may conclude that if $\omega \neq 0$ there exist only three eigenvectors, one null and two space-like, except in the case where $A = 0$ in which there exists a second null eigenvector.

Now consider the case in which σ and ω are both zero, which is the case of the pure radiation field. The expression for $E_\mu{}^\alpha$ is now given by (3.7) and we can see that its eigenvalues are all zero. It possesses three eigenvectors one null and two space-like:

$$V_\mu(1) = l_\mu$$
$$V_\mu(2) = m_\mu + \bar{m}_\mu$$
$$V_\mu(3) = \mathrm{i}(m_\mu - \bar{m}_\mu).$$

The results of this paragraph may be summarized as follows. If a space–time admits a combined neutrino–gravitational field with positive energy density the neutrino flux vector l_μ is an eigenvector of the Ricci tensor and defines a null geodesic congruence. The congruence will have non-zero twist and shear if l_μ corresponds to a double eigenvalue of the Ricci tensor and the tensor has also two space–like eigenvectors corresponding to the remaining two distinct eigenvalues. The congruence will have zero shear and non-zero twist if the Ricci tensor possesses two pairs of double eigenvalues. Finally, the neutrino congruence must be shear-free if it is twist-free and the Ricci tensor must then have the form $R_{\mu\nu} = 2Al_\mu l_\nu$.

These results may also be expressed in terms of Plebanski's (1964) classification of the traceless Ricci tensor. In figure 1 we show the different types of Ricci tensor distinguished by Plebanski which can be interpreted as describing a neutrino field with positive energy density. We give this using Plebanski's notation. The eigenvalues are described in the square brackets. The symbol T, N or S is used to represent an eigenvalue which contains a time-like vector, no time-like vector but a null vector, or only space-like vectors. The numbers sometimes placed before these

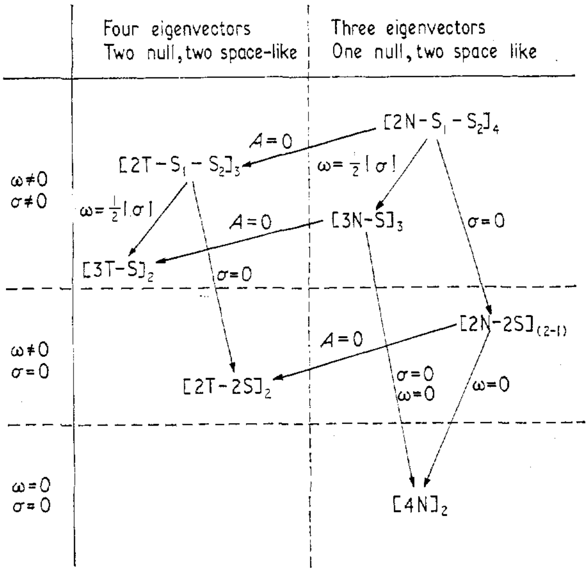

Figure 1. Plebanski classification of the neutrino field.

denote repeated eigenvalues. The numbers outside the square brackets are the indices of nil-potency in the same order in which the eigenvalues are given. Where there is no ambiguity the sum of the indices of nil-potency is given. The arrows indicate degenerations obtained by restrictions on the values of the parameters A, ω and σ. The figure illustrates the fact that, for different values of the parameters, the energy–momentum tensor of the neutrino field with positive energy density can be divided into seven distinct classes. The exceptional case referred to in § 3 in which $B \neq 0$ is of type $[3N-S]_3$ or possibly $[3T-S]_2$.

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References

Griffiths, J. B., and Newing, R. A., 1970, *J. Phys.* A: *Gen. Phys.*, **3,** 269–73.
Inomata, A., and McKinley, W. A., 1965, *Phys. Rev.*, **140,** 1467–73.
Jordan, P., Ehlers, J., and Sachs, R., 1961, *Akad. Wiss. Mainz*, No. 1.
Newman, E., and Penrose, R., 1962, *J. math. Phys.*, **3,** 566–78.
Penney, R., 1965, *J. math. Phys.*, **6,** 1309–14.
Plebanski, J., 1964, *Acta. Phys. Polon.*, **26,** 963–1020.